Approximate interacting solitary wave solutions for a pair of coupled nonlinear Schrödinger equations

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Using a modified Cole-Hopf transformation and the Hirota method for series solutions, approximate interacting solitary wave solutions for a pair of coupled nonlinear Schrödinger equations have been investigated. Previous solutions have been regained. It is noted that if the solution of the first order term satisfied by the Schrödinger equation for a free particle has been taken as a linear superposition of the solutions, then the envelope of the interacting solitary waves are time dependent. If the coupling coefficient is negative, then the amplitudes of the envelope interacting solitary waves are reduced from those in the decoupled limit.

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I. INTRODUCTION

In a plasma, the govering equations for the propagation of the nonlinear wave-wave interaction between the high frequency Langmuir and low frequency ion-acoustic waves may be expressed by a pair of coupled nonlinear Schrödinger equations [1–3]

$$i\frac{\partial\psi_1}{\partial t}+p_1\frac{\partial^2\psi_1}{\partial x^2}=(q_1|\psi_1|^2+q|\psi_2|^2)\psi_1,$$
$$i\frac{\partial\psi_2}{\partial t}+p_2\frac{\partial^2\psi_2}{\partial x^2}=(q_2|\psi_2|^2+q|\psi_1|^2)\psi_2 . \qquad (1)$$

In other cases in plasma, the nonlinear wave-wave interaction such as two transverse waves in a plasma [4–6]; one transverse and a Langmuir waves [6–7], the wave propagations may also be governed by the Eqs. (1). In optics, the model equations for the slowly varying electric field amplitudes in a nonlinear Kerr medium can also be described by the same equations [8–13]. Without loss of generality, the coefficient of cross-phase modulation q may be taken as ± 1, except for the case when the fields are decoupled in which $q=0$. The system (1) may have stable solitary wave solutions [14]. Zakharov and Schulman [15] show the complete integrability of this system in terms "motion invariant" by using the degenerative dispersion laws in the following restricted cases:

$$q_1=q_2=q, \quad p_2=p_1 \qquad (2a)$$

and

$$q_1=q_2=-q, \quad p_2=-p_1 . \qquad (2b)$$

For these cases, the associated Bäcklund transformation and the Hirota bilinearization can be constructed [16]. Recently, interacting solitary wave solutions for the case (2a) were derived by Tratnik and Sipe [17].

In most cases of physical interest these restrictions (2) are not even approximately valid. Therefore, approximate analytical solutions for the evolution of interacting solitary waves are necessary to summarize information and physical insight of the numerical results. In this paper, we wish to investigate approximate solitary wave solutions for the coupled nonlinear Schrödinger equations (1). In Sec. II, we use the Hirota method to derive approximate solitary wave solutions of Eqs. (1). Exact solutions for special cases are given in Sec. III. In Secs. IV and V, approximate and numerical solutions are given, respectively. In the last section, a summary of the paper is given.

II. HIROTA METHOD

The Hirota method [17–19] can be extended to find the approximate solitary wave solutions for the coupled nonlinear Schrödinger equations (1). We can see that Eqs. (1) are invariant under the Galelei transformation [17]

$$x'=x-vt, \quad t'=t ,$$
$$S_j'(x',t')=S_j(x,t)\exp\left[-i\frac{v}{2p_j}\left[x-\frac{v}{2}t\right]\right], \quad j=1,2 , \qquad (3)$$

where $S_1(x,t)$ and $S_2(x,t)$ are the solitary wave solutions of Eqs. (1). So,

$$S_j(x,t)=S_j'(x',t')\exp\left[i\frac{v}{2p_j}(x-\tfrac{1}{2}vt)\right], \quad j=1,2 \qquad (4)$$

are also solitary wave solutions of Eqs. (1) moving with velocity v than the previous solitary waves.

To solve Eqs. (1) we make change of dependent variables from $\psi_1(x,t)$ and $\psi_2(x,t)$ to the functions $f_1(x,t)$, $f_2(x,t)$, $g(x,t)$, and $h(x,t)$ as follows:

$$\psi_1(x,t)=g/f_1, \quad \psi_2(x,t)=h/f_2 , \qquad (5)$$

and

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$$|g|^2/f_1^2=\left[p_1q_2\frac{\partial^2}{\partial x^2}(\ln f_1^2)-p_2q\frac{\partial^2}{\partial x^2}(\ln f_2^2)\right]\Big/(q^2-q_1q_2)\ , \tag{6}$$

$$|h|^2/f_2^2=\left[p_2q_1\frac{\partial^2}{\partial x^2}(\ln f_2^2)-p_1q\frac{\partial^2}{\partial x^2}(\ln f_2^2)\right]\Big/(q^2-q_1q_2)\ , \tag{7}$$

where f_1 and f_2 are assumed as real functions of x and t. Using the transformations (5)–(7) in Eqs. (1), we get

$$i(f_1g_t-gf_{1t})+p_1(f_1g_{xx}-2g_xf_{1x}+gf_{1xx})=0\ , \tag{8}$$

and

$$i(f_2h_t-hf_{2t})+p_2(f_2h_{xx}-2h_xf_{2x}+hf_{2xx})=0\ , \tag{9}$$

where here on the subscripts x and t denote partial differentiation with respect to x and t, respectively. The modified Cole-Hopf transformations in (6) and (7) reduced the pair of coupled nonlinear Schrödinger equations into two homogeneous equations (8) and (9). These equations are decoupled for g and h.

We look for solutions of the Eqs. (8) and (9) in the form [20]

$$\begin{aligned}g&=\varepsilon g^{(1)}+\varepsilon^3g^{(3)}+\cdots\ ,\\ h&=\varepsilon h^{(1)}+\varepsilon^3h^{(3)}+\cdots\ ,\\ f_j&=1+\varepsilon^2f_j^{(2)}+\varepsilon^4f_j^{(4)}+\cdots,\quad \text{for } j=1,2\ .\end{aligned} \tag{10}$$

The factor ε is a convenient small expansion parameter. Substituting (10) into (6)–(9) we deduce relations connecting the different $f_1^{(n)}$, $f_2^{(n)}$, $g^{(n)}$, and $h^{(n)}$ at each order of ε. At first order, one can get

$$ig_t^{(1)}+p_1g_{xx}^{(1)}=0\ , \tag{11}$$

and

$$ih_t^{(1)}+p_2h_{xx}^{(1)}=0\ . \tag{12}$$

The solutions of (11) and (12) may be written as

$$g^{(1)}(x,t)=Y_1+Y_2\ , \tag{13}$$

and

$$h^{(1)}(x,t)=Z_1+Z_2\ , \tag{14}$$

with

$$\begin{aligned}Y_j&=A_j\exp[a_j(x+ip_1a_jt)]\ ,\\ Z_j&=B_j\exp[b_j(x+ip_2b_jt)]\ ,\\ a_j&>0,\quad b_j>0\quad \text{for } j=1,2\ ,\end{aligned} \tag{15}$$

where A_1, A_2, B_1, B_2, a_1, a_2, b_1, and b_2 are arbitrary constants. To second order, in ε we get

$$2p_1f_{1xx}^{(2)}=-q_1|g^{(1)}|^2-q|h^{(1)}|^2\ , \tag{16}$$

and

$$2p_2f_{2xx}^{(2)}=-q_2|h^{(1)}|^2-q|g^{(1)}|^2\ . \tag{17}$$

Using (13)–(15) in (16) and (17), we get after integration

$$\begin{aligned}f_1^{(2)}=&-\frac{q_1}{2p_1}\left[\frac{|Y_1|^2}{4a_1^2}+\frac{|Y_2|^2}{4a_2^2}+\frac{2|Y_1Y_2|}{(a_1+a_2)^2}\cos[p_1(a_1^2-a_2^2)t]\right]\\ &-\frac{q}{2p_1}\left[\frac{|Z_1|^2}{4b_1^2}+\frac{|Z_2|^2}{4b_2^2}+\frac{2|Z_1Z_2|}{(b_1+b_2)^2}\cos[p_2(b_1^2-b_2^2)t]\right]\end{aligned} \tag{18}$$

and

$$\begin{aligned}f_2^{(2)}=&-\frac{q_2}{2p_2}\left[\frac{|Z_1|^2}{4b_1^2}+\frac{|Z_2|^2}{4b_2^2}+\frac{2|Z_1Z_2|}{(b_1+b_2)^2}\cos[p_2(b_1^2-b_2^2)t]\right]\\ &-\frac{q}{2p_2}\left[\frac{|Y_1|^2}{4a_1^2}+\frac{|Y_2|^2}{4a_2^2}+\frac{2|Y_1Y_2|}{(a_1+a_2)^2}\cos[p_1(a_1^2-a_2^2)t]\right]\ .\end{aligned} \tag{19}$$

To the third order in ε, we get

$$ig_t^{(3)}+p_1g_{xx}^{(3)}=ig^{(1)}f_{1t}^{(2)}+2p_1g_x^{(1)}f_{1x}^{(2)}-p_1g^{(1)}f_{1xx}^{(2)} \tag{20}$$

and

$$ih_t^{(3)}+p_2h_{xx}^{(3)}=ih^{(1)}f_{2t}^{(2)}+2p_2h_x^{(1)}f_{2x}^{(2)}-p_2h^{(1)}f_{2xx}^{(2)}\ . \tag{21}$$

Using (13)–(15), (18), and (19) in (20) and (21), we may get the solutions of (20) and (21) in the forms

$$\begin{aligned}g^{(3)}=&-\frac{q_1}{8p_1}\left[\frac{a_1-a_2}{a_1+a_2}\right]^2\left[\frac{|Y_1|^2Y_2}{a_1^2}+\frac{|Y_2|^2Y_1}{a_2^2}\right]\\ &+q[\{Y_1F(p_1,p_2,a_1,b_1,b_2)+Y_2F(p_1,p_2,a_2,b_1,b_2)\}Z_1Z_2^*+(b_1\leftrightarrow b_2,\ Z_1\leftrightarrow Z_2)]\\ &+\frac{q}{8p_1}\left[\frac{1}{b_1^2}\left\{\frac{b_1-a_1}{a_1+b_1}Y_1+\frac{b_1-a_2}{a_2+b_1}Y_2\right\}|Z_1|^2+(b_1\rightarrow b_z,\ Z_1\rightarrow Z_2)\right]\end{aligned} \tag{22}$$

and

$$h^{(3)}=-\frac{q_2}{8p_2}\left[\frac{b_1-b_2}{b_1+b_2}\right]^2\left[\frac{|Z_1|^2Z_2}{b_1^2}+\frac{|Z_2|^2Z_1}{b_2^2}\right]$$
$$+q[\{Z_1F(p_2,p_1,b_1,a_1,a_2)+Z_2F(p_2,p_1,b_2,a_1,a_2)\}Y_1Y_2^*+(a_1\leftrightarrow a_2,\ Y_1\leftrightarrow Y_2)]$$
$$+\frac{q}{8p_2}\left[\frac{1}{a_1^2}\left\{\frac{a_1-b_1}{a_1+b_1}Z_1+\frac{a_1-b_2}{a_1+b_2}Z_2\right\}|Y_1|^2+(a_1\to a_2,\ Y_1\to Y_2)\right], \tag{23}$$

where we define $F(p_1,p_2,a,b,c)$ as

$$F(p_1,p_2,a,b,c)=\frac{2p_2(b-c)+p_1(b+c-2a)}{2p_1(b+c)[p_1(a+b+c)^2-(p_1a^2+p_2b^2-p_2c^2)]}. \tag{24}$$

To the fourth order of ε, we may get in a similar way

$$f_1^{(4)}=\frac{q_1^2}{64p_1^2a_1^2a_2^2}\left[\frac{a_1-a_2}{a_1+a_2}\right]^4|Y_1Y_2|^2-\frac{q(p_1q_2-p_2q}{8p_1^2p_2}\sum_{i,j=1}^{2}\frac{Z_i^{*2}Z_j^2}{(b_i+b_j)^4}$$
$$+\frac{q}{8p_1(b_1+b_2)^2}-\left[\frac{q(b_1+b_2)^2}{8p_1b_1^2b_2^2}+\frac{2q}{p_1(b_1+b_2)^2}-\frac{q_2}{p_2b_1b_2}\right]|Z_1Z_2|^2$$
$$-\frac{q(p_1q_2-p_2q)}{16p_1^2p_1}\sum_{\substack{i,j=1\\ i\neq j}}^{2}\frac{(Z_1^*Z_j+Z_iZ_j^*)|Z_j|^2}{b_j^2(b_i+b_j)^2}-\sum_{i,j,k,l=1}^{2}\frac{\lambda_{ijkl}^{(1)}Y_i^*Y_jZ_kZ_l^*}{2p_1(a_i+a_j+b_k+k_l)^2} \tag{25}$$

and

$$f_2^{(4)}=\frac{q_2^2(b_1-b_2)^4}{64p_2^2b_1^2b_2^2(b_1+b_2)^4}|Z_1Z_2|^2-\frac{q(p_2q_1-p_1q)}{8p_1p_2^2}\sum_{i,j=1}^{2}\frac{Y_1^{*2}Y_j^2}{(a_i+a_j)^2}$$
$$+\frac{q}{8p_2(a_1+a_2)^2}\left[\frac{q(a_1+a_2)^2}{8p_2a_1^2a_2^2}+\frac{2q}{p_2(a_1+a_2)^2}-\frac{q_1}{p_1a_1a_2}\right]|Y_1Y_2|^2$$
$$-\frac{q(p_2q_1-p_1q)}{16p_1p_2^2}\sum_{\substack{i,j=1\\ i\neq j}}^{2}\frac{(Y_i^*Y_j+Y_iY_j^*)|Y_j|^2}{a_j^2(a_i+a_j)^2}-\sum_{i,j,k,l=1}^{2}\frac{\lambda_{ijkl}^{(2)}Z_i^*Z_jY_kY_l^*}{2p_2(b_i+b_j+a_k+a_l)^2}, \tag{26}$$

where

$$\lambda_{ijkl}^{(1)}=-\frac{qq_1}{p_1(a_i+a_j)(b_k+b_l)}+\frac{qq_1}{2p_1(b_k+b_l)^2}-\frac{qq_1}{2p_1(a_i+a_j)^2}+\frac{q^2}{p_2(a_i+a_j)^2}$$
$$+qq_1\{F(p_1,p_2,a_j,b_k,b_l)+F(p_1,p_2,a_i,b_l,b_k)\}+q^2\{F(p_2,p_1,b_k,a_j,a_i)+F(p_2,p_1,b_l,a_i,a_j)\} \tag{27}$$

and

$$\lambda_{ijkl}^{(2)}=-\frac{qq_2}{p_2(b_i+b_j)(a_k+a_l)}+\frac{qq_2}{2p_2(a_k+a_l)^2}-\frac{qq_2}{2p_2(b_i+b_j)^2}+\frac{q^2}{p_1(b_i+b_j)^2}$$
$$+qq_2\{F(p_2,p_1,b_j,a_k,a_l)+F(p_2,p_1,b_i,a_l,a_k)\}+q^2\{F(p_1,p_2,a_k,b_j,b_i)+F(p_i,p_2,a_l,b_i,b_j)\}. \tag{28}$$

In the same way, all other order in ε can also be calculated. Solutions of Eqs. (1) then become

$$\psi_1 = \frac{\varepsilon g^{(1)} + \varepsilon^3 g^{(3)} + \cdots}{1 + \varepsilon^2 f_1^{(2)} + \varepsilon^4 f_1^{(4)} + \cdots} \tag{29}$$

and

$$\psi_2 = \frac{\varepsilon h^{(1)} + \varepsilon^{(3)} h^{(3)} + \cdots}{1 + \varepsilon^2 f_2^{(2)} + \varepsilon^4 f_2^{(4)} + \cdots} . \tag{30}$$

Since the terms $g^{(2j-1)}$, $h^{(2j-1)}$, and $f_{1,2}^{(2j)}$ are homogeneous functions of Y_1, Y_2, Z_1, and Z_2 of degrees $2j-1$, $2j-1$, and $2j$, respectively, so Eqs. (15) suggest that for the convergence of our solutions (29) and (30), one should truncate the series, if required, after an even order of ε.

III. SPECIAL SOLUTIONS

In general, for arbitrary p_1, p_2, q_1, q_2, and q the solutions (29) and (30) will not be exact. However, we can get exact solutions in special cases. First of all, we consider the decoupled system for (1), i.e., for $q=0$, in which we may find that

$$\begin{bmatrix} g^{(2n-1)} \\ h^{(2n-1)} \\ f_1^{(2n)} \\ f_2^{(2n)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{for } n \geq 3 . \tag{31}$$

Then the solution (29) becomes exact

$$\psi_1(x,t) = \sqrt{-2p_1/q_1}\,\frac{a_1+a_2}{|a_1-a_2|} \times \left[a_1 \operatorname{sech}\left[a_1 x + \ln\left|\frac{a_1-a_2}{a_1+a_2}\right|\right] \times \exp(ip_1 a_1^2 t) + a_2 \operatorname{sech}\left[a_2 x + \ln\left|\frac{a_1-a_2}{a_1+a_2}\right|\right] \times \exp(ip_1 a_2^2 t)\right] \Bigg/ \left[1 + \frac{4a_1 a_2}{(a_1+a_2)^2} \times \frac{\{\exp(2a_1 x) + \exp(2a_2 x) + 2\exp(a_1 x + a_2 x)\cos[p_1(a_1^2 - a_2^2)t]\}}{\left\{1 + \left[\frac{a_1-a_2}{a_1+a_2}\right]^2 \exp(2a_1 x)\right\}\left\{1 + \left[\frac{a_1-a_2}{a_1+a_2}\right]^2 \exp(2a_2 x)\right\}}\right] , \tag{32}$$

where we set $\varepsilon = 1$ and

$$A_1 = 2a_1\sqrt{-2p_1/q} ; \quad A_2 = 2a_2\sqrt{-2p_2/q_1} . \tag{33}$$

Similar solution for $\psi_2(x,t)$ may be obtained.

Next, let us take $f_1 = f_2 = f$ and solutions of (11) and (12) as

$$g^{(1)}(x,t) = 2a\left[2\frac{p_1 q_2 - p_2 q}{q^2 - q_1 q_2}\right]^{1/2} \exp[a(x + ip_1 a t)] , \tag{34}$$

and

$$h^{(1)}(x,t) = 2a\left[2\frac{p_2 q_1 - p_1 q}{q^2 - q_1 q_2}\right]^{1/2} \exp[a(x + ip_2 a t)] , \tag{35}$$

provided

$$(p_1 q_2 - p_2 q)/(q^2 - q_1 q_2) > 0 ,$$

and

$$(p_2 q_1 - p_1 q)/(q^2 - q_1 q_2) > 0 .$$

In this case, we may get from (18) and (19)

$$f^{(2)} = \exp(2ax) . \tag{36}$$

Using (34)–(36) in (5)–(9), we may find

$$\begin{bmatrix} g^{(2n-1)} \\ h^{(2n-1)} \\ f^{(2n)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{for } n \geq 2 . \tag{37}$$

Then the solutions (29) and (30) become [20] as

$$\psi_1(x,t) = 2a\left[2\frac{p_1 q_2 - p_2 q}{q^2 - q_1 q_2}\right]^{1/2} \operatorname{sech} ax \exp(ip_1 a^2 t) , \tag{38}$$

$$\psi_2(x,t) = 2a\left[2\frac{p_2 q_1 - p_1 q}{q^2 - q_1 q_2}\right]^{1/2} \operatorname{sech} ax \exp(ip_2 a^2 t) , \tag{39}$$

where we have again used $\varepsilon = 1$.

Finally, we take the problem considered by Tratnik and Sipe [17]. In this case, we have

$$p_1 = p_2 = 1, \quad q_1 = q_2 = -1 , \tag{40}$$

and

$$q \to -1 .$$

Using (40) in (5)–(7), we get their transformations (3.4)

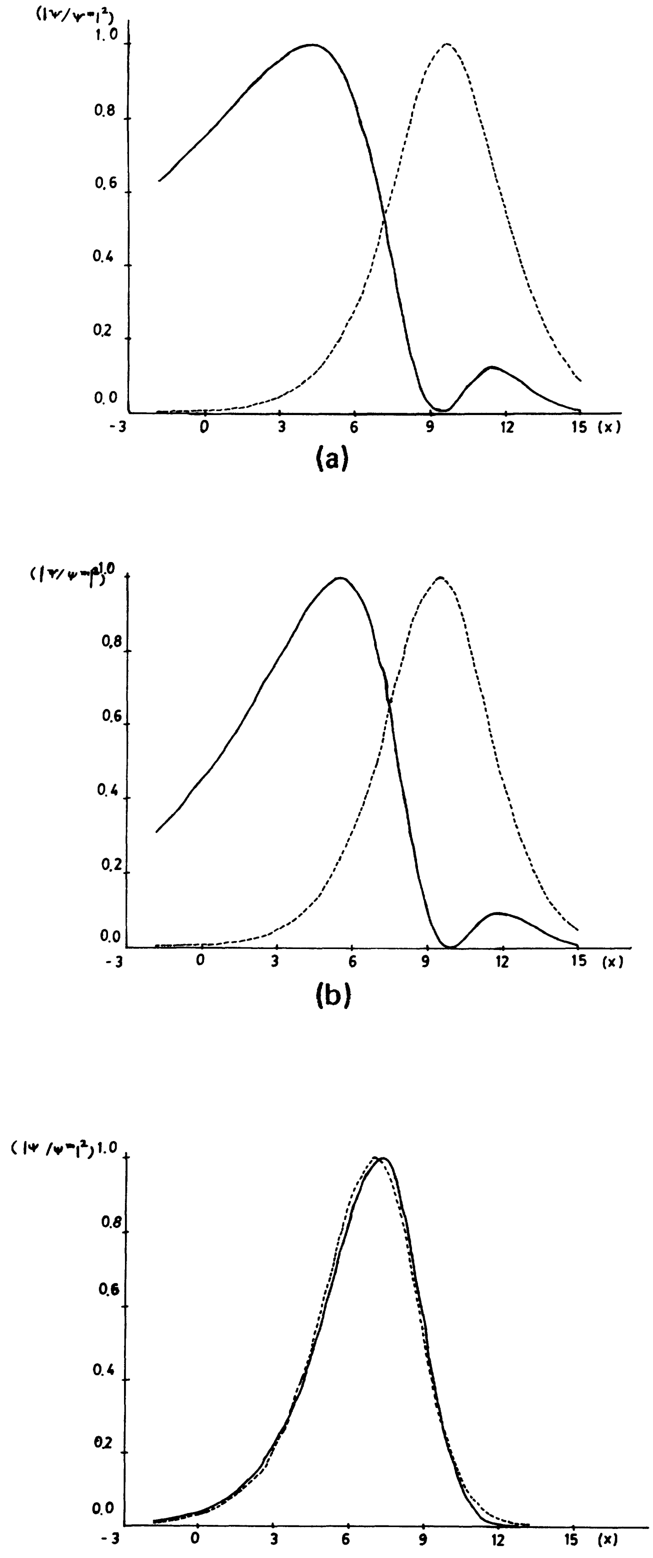

FIG. 1. $|\psi/\psi^{(m)}|^2$ is plotted against x for the solutions (48) for $p_1=0.45$, $p_2=0.55$, $q_1=-0.45$, $q_2=-\frac{5}{9}$, $q=-1$, $\varepsilon=0.05$, $x_{10}=0.9$, $x'_{10}=0.9$, $a_1=\frac{1}{3}$, (a) $b_1=0.05$, $|\psi_1^{(m)}|^2=0.196$, $|\psi_2^{(m)}|^2=7.02\times10^{-5}$: ——, $|\psi_1/\psi_1^{(m)}|^2$; — — —, $|\psi_2/\psi_2^{(m)}|^2$; (b) $b_1=0.10$, $|\psi_1^{(m)}|^2=0.174$, $|\psi_2^{(m)}|^2=5.14\times10^{-4}$: — — —, $|\psi_1/\psi_1^{(m)}|^2$; ——, $|\psi_2/\psi_2^{(m)}|^2$; (c) $b_1=0.30$, $|\psi_1^{(m)}|^2=3.59\times10^{-2}$, $|\psi_2^{(m)}|^2=7.56\times10^{-2}$: — — —, $|\psi_1/\psi_1^{(m)}|^2$; ——, $|\psi_2/\psi_2^{(m)}|^2$.

and (3.5). For the solutions parameters, we should take the solutions (13)–(15) as

$$g^{(1)}=Y_1,\quad h^{(1)}=Z_1$$

with (41)

$$A_1=2\sqrt{2}a_1\exp(-a_1x_{10});\quad B_1=2\sqrt{2}b_1\exp(-b_1x_{20})\ .$$

Using (40) and (41) in (18), (19), (22), (23), (25), and (26) we may get

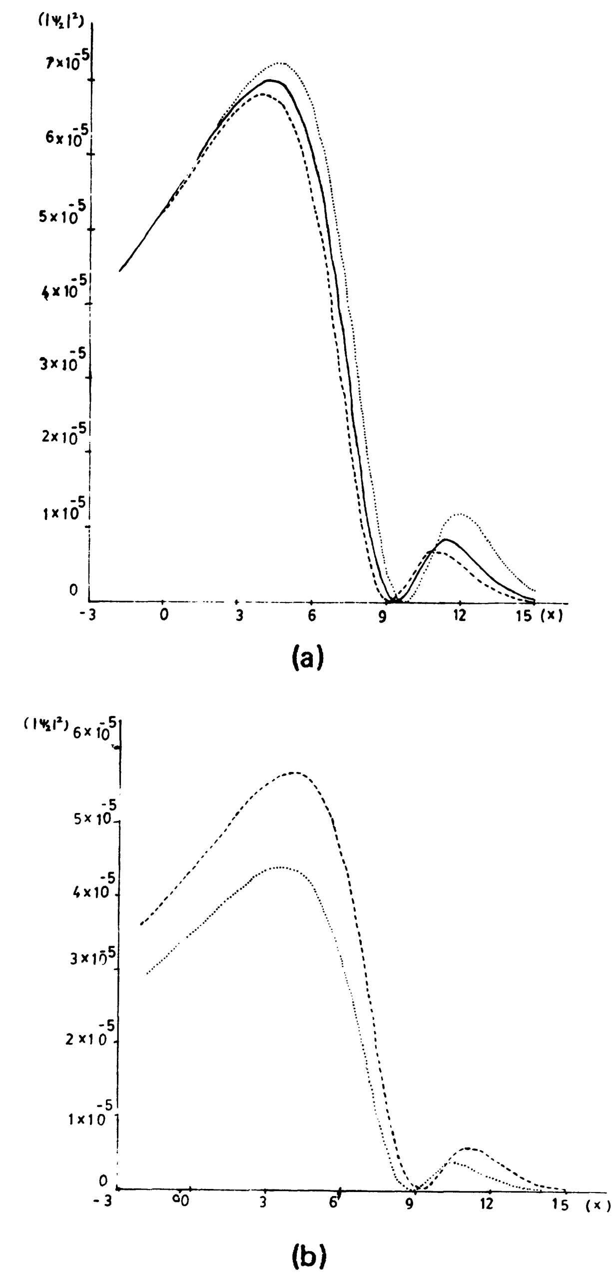

FIG. 2. $|\psi_2|^2$ is plotted against x for the solutions (48) for $\varepsilon=0.05$, $x_{10}=0.9$, $x'_{10}=0$, $q_1=-0.45$, $q_2=-\frac{5}{9}$, $q=-1$, $a_1=\frac{1}{3}$, $b_1=0.03$, (a) $p_2=0.6$: ——, $p_1=0.5$; — — —, $p_1=0.6$; . . . ., $p_1=0.4$; (b) $b_1=0.5$: — — —, $p_2=0.5$; . . . ., $p_2=0.4$.

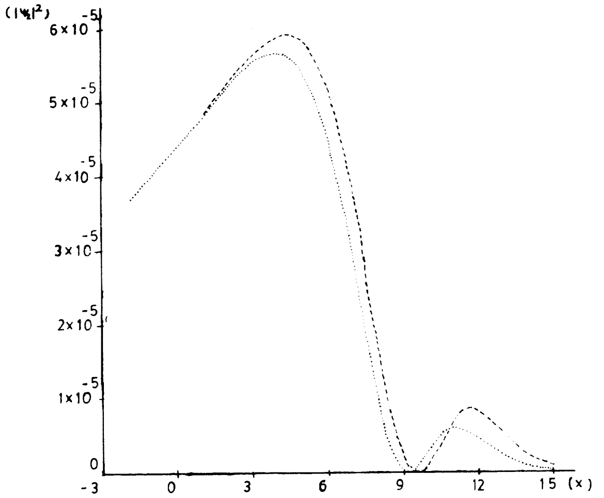

FIG. 3. $|\psi_2|^2$ is plotted against x for the solutions (48) for $\varepsilon=0.05$, $x_{10}=0.9$, $x'_{10}=0.0$, $a_1=\frac{1}{3}$, $b_1=0.05$, $p_1=0.5$, $p_2=0.45$, $q_2=-\frac{5}{9}$, $q=-1$: . . . ., $q_1=-0.45$; — — —, $q_1=-0.6$.

$$f_1^{(2)}=f_2^{(2)}=\exp[2a_1(x-x_{10})]+\exp[2b_1(x-x_{20})] , \tag{42}$$

$$g^{(3)}=2\sqrt{2}a_1\left[\frac{a_1-b_1}{a_1+b_1}\right]\times\exp[a_1(x-x_{10})+2b_1(x-x_{20})]\exp(ia_1^2t) , \tag{43a}$$

$$h^{(3)}=-2\sqrt{2}b_1\left[\frac{a_1-b_1}{a_1+b_1}\right]\times\exp[2a_1(x-x_{10})+b_1(x-x_{20})]\exp(ib_1^2t) , \tag{43b}$$

$$f_1^{(4)}=f_2^{(4)}=\left[\frac{a_1-b_1}{a_1+b_1}\right]^2\exp[2a_1(x-x_{10})+2b_1(x-x_{20})] , \tag{44}$$

and

$$\begin{bmatrix} g^{(2n-1)} \\ h^{(2n-1)} \\ f_1^{(2n)} \\ f_2^{(2n)} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{for } n\geq 3 . \tag{45}$$

Then the solutions ψ_1 and ψ_2 may be expressed as

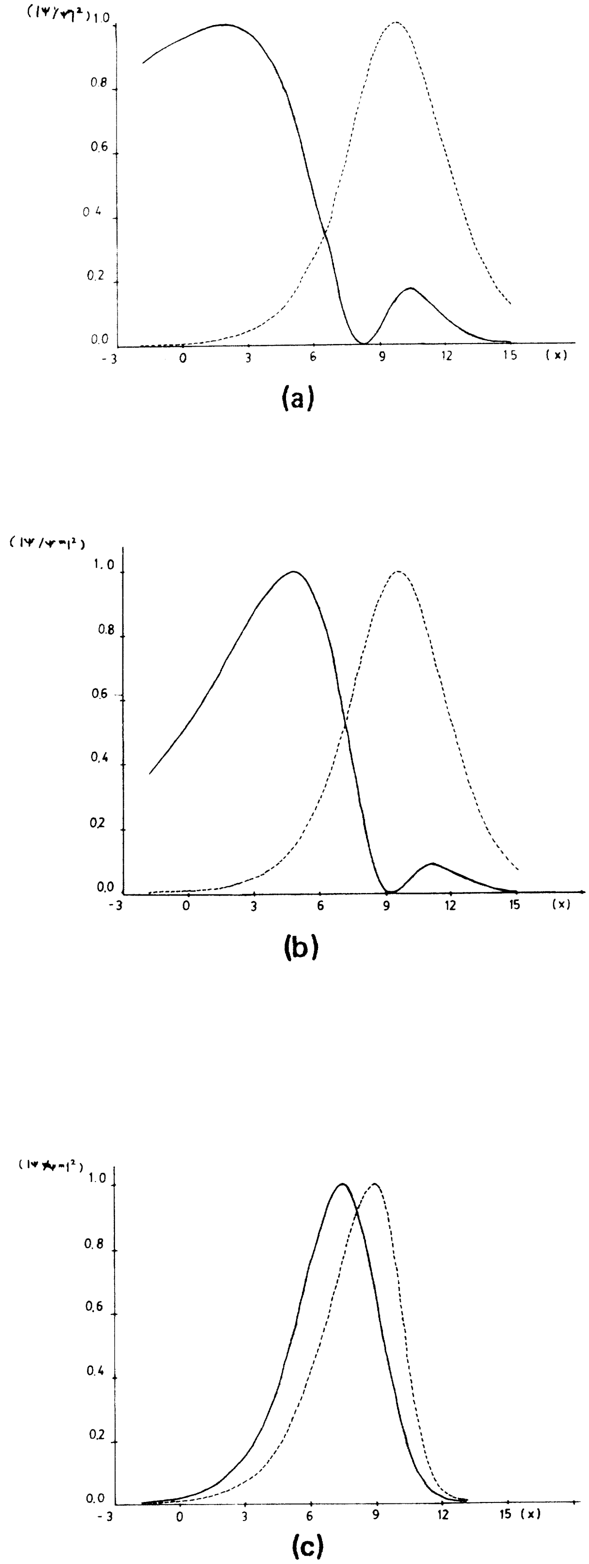

FIG. 4. $|\psi/\psi^{(m)}|^2$ is plotted against x when the solutions of $g^{(1)}$ and $h^{(1)}$ are taken as in (50) for $\varepsilon=0.05$, $x_{10}=0.9$, $x'_{10}=0.9$, $p_1=0.5$, $p_2=0.45$, $q_1=-0.45$, $q_2=-\frac{5}{9}$, $t=0$, $a_1=\frac{1}{3}$, $a_2=0.03$, (a) $b_1=0.03$, $|\psi_1^{(m)}|^2=0.227$, $|\psi_2^{(m)}|^2=1.33\times10^{-5}$: — — —, $|\psi_1/\psi_1^{(m)}|^2$; ——, $|\psi_2/\psi_2^{(m)}|^2$, (b) $b_1=0.1$, $|\psi_1^{(m)}|^2=0.213$, $|\psi_2^{(m)}|^2=2.45\times10^{-4}$: — — —, $|\psi_1/\psi_1^{(m)}|^2$; ——, $|\psi_2/\psi_2^{(m)}|^2$, (c) $b_1=\frac{1}{3}$, $|\psi_1^{(m)}|^2=0.133$, $|\psi_2^{(m)}|^2=4.21\times10^{-2}$: — — —, $|\psi_1/\psi_1^{(m)}|^2$; ——, $|\psi_2/\psi_2^{(m)}|^2$.

$$\psi_1(x,t)=2\sqrt{2}a_1\exp[a_1(x-x_{10})+ia_1^2t]$$
$$\times\left\{1+\frac{a_1-b_1}{a_1+b_1}\exp[2b_1(x-x_{20})]\right\}\Bigg/\left\{1+\exp[2a_1(x-x_{10})]+\exp[2b_1(x-x_{20})]+\left[\frac{a_1-b_1}{a_1+b_1}\right]^2\exp[2a_1(x-x_{10})+2b_1(x-x_{20})]\right\}, \tag{46a}$$

$$\psi_2(x,t)=2\sqrt{2}b_1\exp[b_1(x-x_{20})+ib_1^2t]$$
$$\times\left\{1-\frac{a_1-b_1}{a_1+b_1}\exp[2a_1(x-x_{10})]\right\}\Bigg/\left\{1+\exp[2a_1(x-x_{10})]+\exp[2b_1(x-x_{20})]+\left[\frac{a_1-b_1}{a_1+b_1}\right]^2\exp[2a_1(x-x_{10})+2b_1(x-x_{20})]\right\}, \tag{46b}$$

where we used $\varepsilon=1$. These exact interacting solitary wave solutions (46) correspond to the solutions of Tratnik and Sipe [17]. The properties of these interacting solitary waves were discussed in that paper.

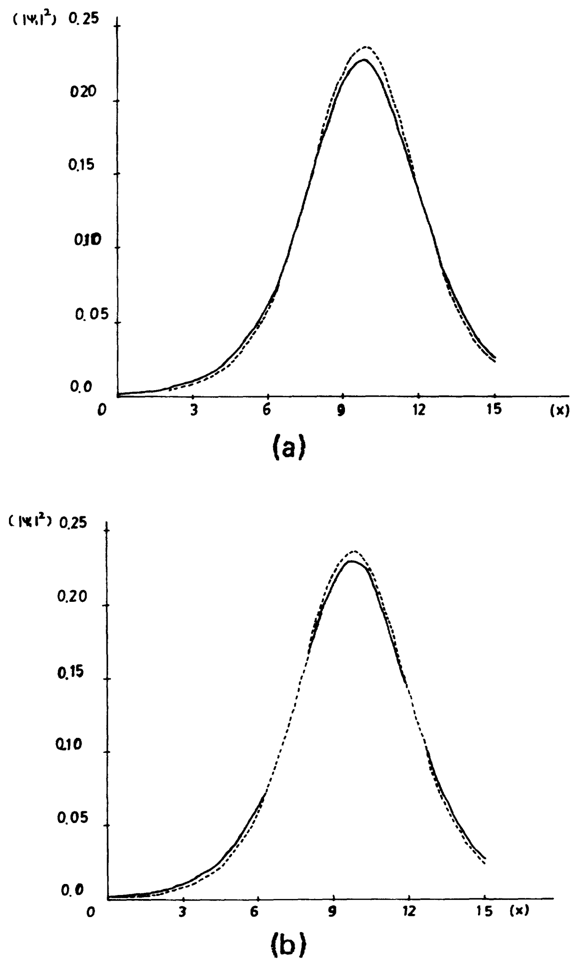

FIG. 5. $|\psi_1|^2$ is plotted against x when the solutions of $g^{(1)}$ and $h^{(1)}$ are taken as in (50) for $\varepsilon=0.05$, $x_{10}=0.9$, $x'_{10}=0.9$, $p_1=0.5$, $p_2=0.45$, $q_1=-0.45$, $q_2=-\frac{5}{9}$, $q=-1$, $a_1=\frac{1}{3}$, $a_2=0.03$, $b_1=0.03$: (a) ——, $t=0$; – – –, $t=65$, (b) – – –, $t=65$; ——, $t=130$.

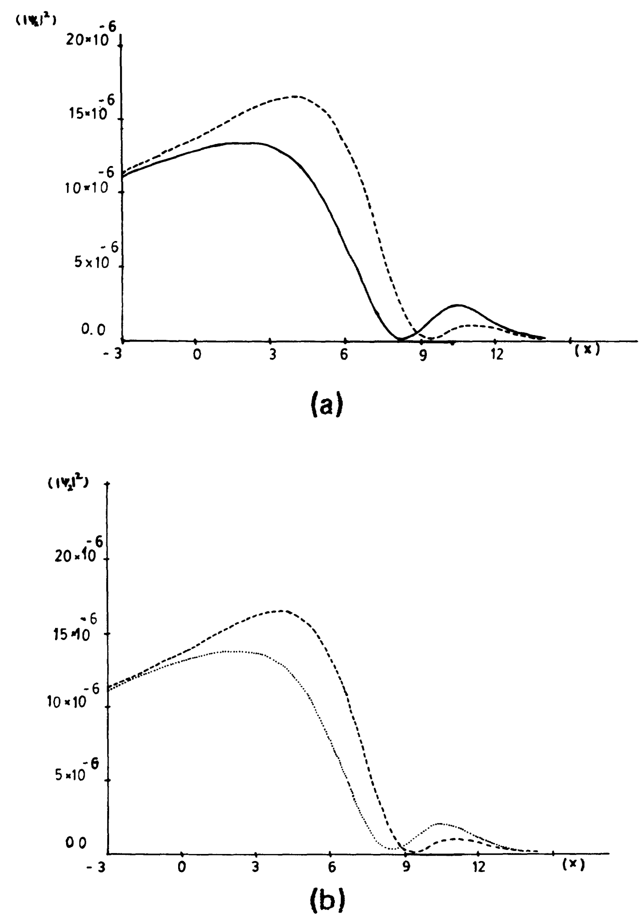

FIG. 6. $|\psi_2|^2$ is plotted against x when the solutions of $g^{(1)}$ and $h^{(1)}$ are taken as in (50) for $\varepsilon=0.05$, $x_{10}=0.9$, $b_1=0.03$, $x'_{10}=0.9$, $p_1=0.5$, $p_2=0.45$, $q_1=-0.45$, $q_2=-\frac{5}{9}$, $q=-1$, $a_1=\frac{1}{3}$, $a_2=0.03$: (a) ——, $t=0$; – – –, $t=65$, (b) – – –, $t=65$; $\cdot\cdot\cdot\cdot$, $t=130$.

IV. APPROXIMATE SOLUTIONS

In this section, the exact solutions as given in (46) have been extended for the case when conditions (40) may be relaxed. In general, for arbitrary values of p_1, p_2, q_1, q_2, and q the solutions (29) and (30) may not terminate. Therefore, truncation of the series at a certain stage is required to get an approximate solution. In our solutions, the term each other of ε involves some powers in exponential of x, so domain x depends on the expansion parameter ε for which our solutions will be accurate. If $g^{(1)}$ and $h^{(1)}$ are taken in the following form

$$g^{(1)}=Y_1 \text{ and } h^{(1)}=Z_1 , \tag{47}$$

then up to the fourth order in ε we can get from (29) and (30)

$$\psi_1(x,t)=2\sqrt{-2p_1/q_1}\,a_1 \exp(\theta+ip_1a_1^2t) \times\left[1+\frac{p_2q(b_1-a_1)}{p_1q_2(a_1+b_1)}\exp(2\theta')\right]\Bigg/\left\{1+\exp(2\theta)+\frac{p_2q}{p_1q_2}\exp(2\theta')\left[1-\frac{p_1q_2-p_2q}{2p_1q_2}\exp(2\theta')\right] -\frac{8p_2a_1^2b_1^2\lambda_{1111}^{(1)}}{q_1q_2(a_1+b_1)^2}\exp[2(\theta+\theta')]\right\}, \tag{48a}$$

and

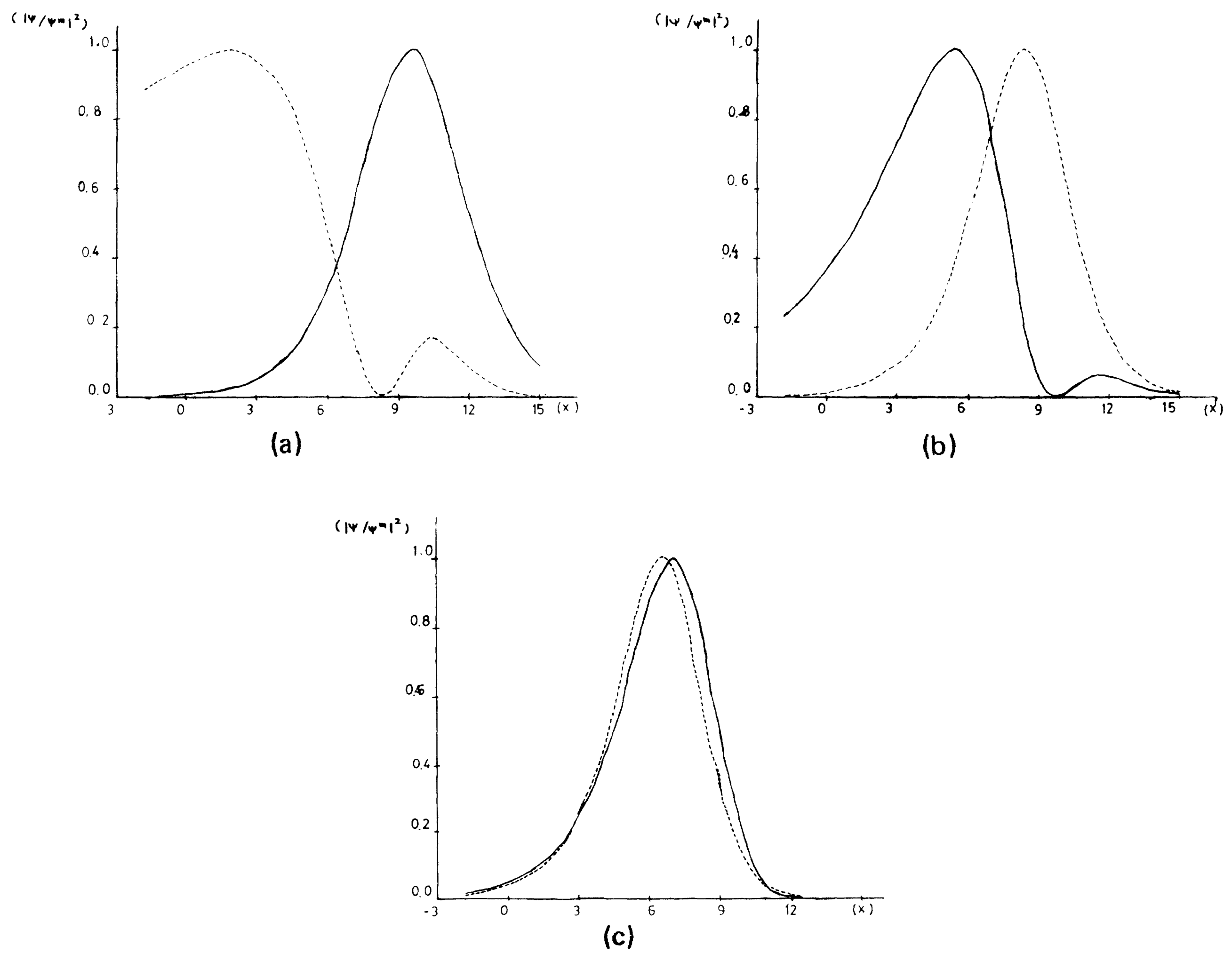

FIG. 7. $|\psi/\psi^{(m)}|^2$ is plotted against x for the solutions (29) and (30) for $\varepsilon=0.05$, $x_{10}=0.9$, $x'_{10}=0.0$, $p_1=0.5$, $p_2=0.45$, $q_1=-0.45$, $q_2=-\frac{5}{9}$, $q=-1$, $a_1=\frac{1}{3}$, $a_2=0.03$, $b_1=0.03$, $t=0$, (a) $b_2=0.03$, $|\psi_1^{(m)}|^2=0.192$, $|\psi_2^{(m)}|^2=5.55\times10^{-5}$: ——, $|\psi_1/\psi_1^{(m)}|^2$; — — —, $|\psi_2/\psi_2^{(m)}|^2$, (b) $b_2=0.15$, $|\psi_1^{(m)}|^2=0.109$, $|\psi_2^{(m)}|^2=1.36\times10^{-3}$: — — —, $|\psi_1/\psi_1^{(m)}|^2$; ——, $|\psi_2/\psi_2^{(m)}|^2$, (c) $b_2=0.30$, $|\psi_1^{(m)}|^2=3.91\times10^{-2}$, $|\psi_2^{(m)}|^2=3.45\times10^{-2}$: — — —, $|\psi_1/\psi_1^{(m)}|^2$; ——, $|\psi_2/\psi_2^{(m)}|^2$.

$$\psi_2(x,t)=2\sqrt{-2p_2/q_2}b_1\exp(\theta'+ip_2b_1^2t)$$
$$\times\left[1+\frac{p_1q(a_1-b_1)}{p_2q_1(a_1+b_1)}\exp(2\theta)\right]\Bigg/\left\{1+\exp(2\theta')+\frac{p_1q}{p_2q_1}\exp(2\theta)\left[1-\frac{p_2q_1-p_1q}{2p_2q_1}\exp(2\theta)\right]\right.$$
$$\left.-\frac{8p_1a_1^2b_1^2\lambda_{1111}^{(2)}}{q_1q_2(a_1+b_1)^2}\exp[2(\theta+\theta')]\right\}. \tag{48b}$$

To get (48) we have used

$$A_1=2a_1\sqrt{-2p_1/q_1}\exp(-a_1x_{10}),$$
$$B_1=2b_1\sqrt{-2p_2/q_2}\exp(-b_1x'_{10}), \tag{49}$$
$$\varepsilon=\exp(-a_1\delta_1)=\exp(-b_1\delta'_1),$$

and

$$\theta=a_1(x-x_{10}-\delta_1);\quad \theta'=b_1(x-x'_{10}-\delta'_1).$$

If the value of p_1, p_2, q_1, q_2, and q are used as in (40) and x'_{10} is replaced by x_{20}, then solutions (48) reduced to (46) for $\delta_1=0=\delta'_1$. The solutions (48) show that $|\psi_1|$ and $|\psi_2|$ are time-independent pulses. If the system (1) is single decoupled, i.e., $q=0$, then (48) gives either

$$\psi_1=\sqrt{-2p_1/q_1}a_1\exp(ip_1a_1^2t)\text{sech}[a_1(x-x_{10}-\delta_1)],$$
$$\psi_2=0; \tag{50a}$$

or

$$\psi_1=0,$$
$$\psi_2=\sqrt{-2p_2/q_2}b_1\exp(ip_2b_1^2t)\text{sech}[b_1(x-x'_{10}-\delta'_2)] \tag{50b}$$

as expected.

To interpret Eqs. (48), let us take $|h^{(1)}|\ll 1$ with $|g^{(1)}|$ as finite. In this case Eqs. (48) reduce to (50a), which is the single decoupled soliton solution where a_1 is related to the amplitude and x_{10} to the position of the maximum. If we take the limit $|g^{(1)}|\ll 1$ with $|h^{(1)}|$ as finite then Eqs. (48) give the other soliton solution (50b), where b_1 is related to the amplitude and x'_{10} to the position of the maximum. In the case where either $g^{(1)}$ or $h^{(1)}$ or both are large, we get $\psi_1\approx 0$ and $\psi_2\approx 0$. When both $h^{(1)}$ and $g^{(1)}$ are finite, Eqs. (48) give the interacting solutions. Equations (48) represent well behaved solutions that tends to zero as magnitude of x tends to infinite.

V. NUMERICAL SOLUTIONS

For our numerical computation we set

$$A_j=2a_j\sqrt{-2p_1/q_1}\exp(-a_jx_{j0}),$$
$$B_j=2b_j\sqrt{-2p_2/q_2}\exp(-b_jx'_{j0}),$$
$$x_{20}=-x_{10}\ \text{and}\ x'_{20}=-x'_{10}.$$

Moreover, we have taken $p_1>0$, $p_2>0$, and $q<0$ because the solitary wave solutions (38) and (39) for the Eqs. (1) are stable in this case [14]. The values of q_1 and q_2 are taken as negative to ensure that in the absence of coupling we may regain the solitary wave solution (50). Solution (48) are shown in Figs. 1–3. Comparing solutions (50) with Fig. 1 we may find that when $a_1\approx b_1$, the shape of the envelope solitary waves ψ_1 and ψ_2 remain almost unaltered but their maxima are reduced due to coupling. If $a_1\gg b_1$, then the envelope of the solitary wave ψ_1 remains almost unaffected due to coupling with ψ_2, on the other hand, the envelope of the solitary wave ψ_2 is deformed due to coupling with ψ_1. Figures 2 and 3 show the effects of the change of dispersion and cubic nonlinear coefficients on the envelope solitary wave respectively. Figures 4–6 show the solutions ψ_1 and ψ_2 when $g^{(1)}$ and $h^{(1)}$ are taken in the following forms

$$g^{(1)}(x,t)=Y_1+Y_2,\quad h^{(1)}(x,t)=Z_1. \tag{51}$$

From Fig. 4, we may see that $|\psi_1|$ and $|\psi_2|$ have the same property as in Fig. 1 if we take $g^{(1)}$ and $h^{(1)}$ in (47) instead of (51). Figures 5 and 6 show that $|\psi_1|$ and $|\psi_2|$ are time dependent and the evolution of the pulses are periodic.

Finally, we have taken the solutions $g^{(1)}$ and $h^{(1)}$ as in (13)–(15). These are plotted in Figs. 7–10. Effects of nonlinearity on the evolution of the pulses are shown in Fig. 8. Evolution of the pulses and the effects of the

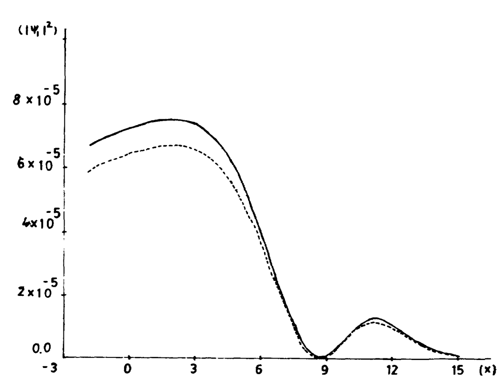

FIG. 8. $|\psi_1|^2$ is plotted against x for the solutions (29) for $\varepsilon=0.05$, $x'_{10}=0.9$, $x'_{10}=0$, $p_1=0.5$, $p_2=0.55$, $q_2=-\frac{5}{9}$, $q=-1$, $a_1=0.03$, $a_2=0.03$, $b_1=0.03$, $b_2=0.3$, $t=0$: ——, $q_1=-0.45$; · · · ·, $q_1=-0.51$.

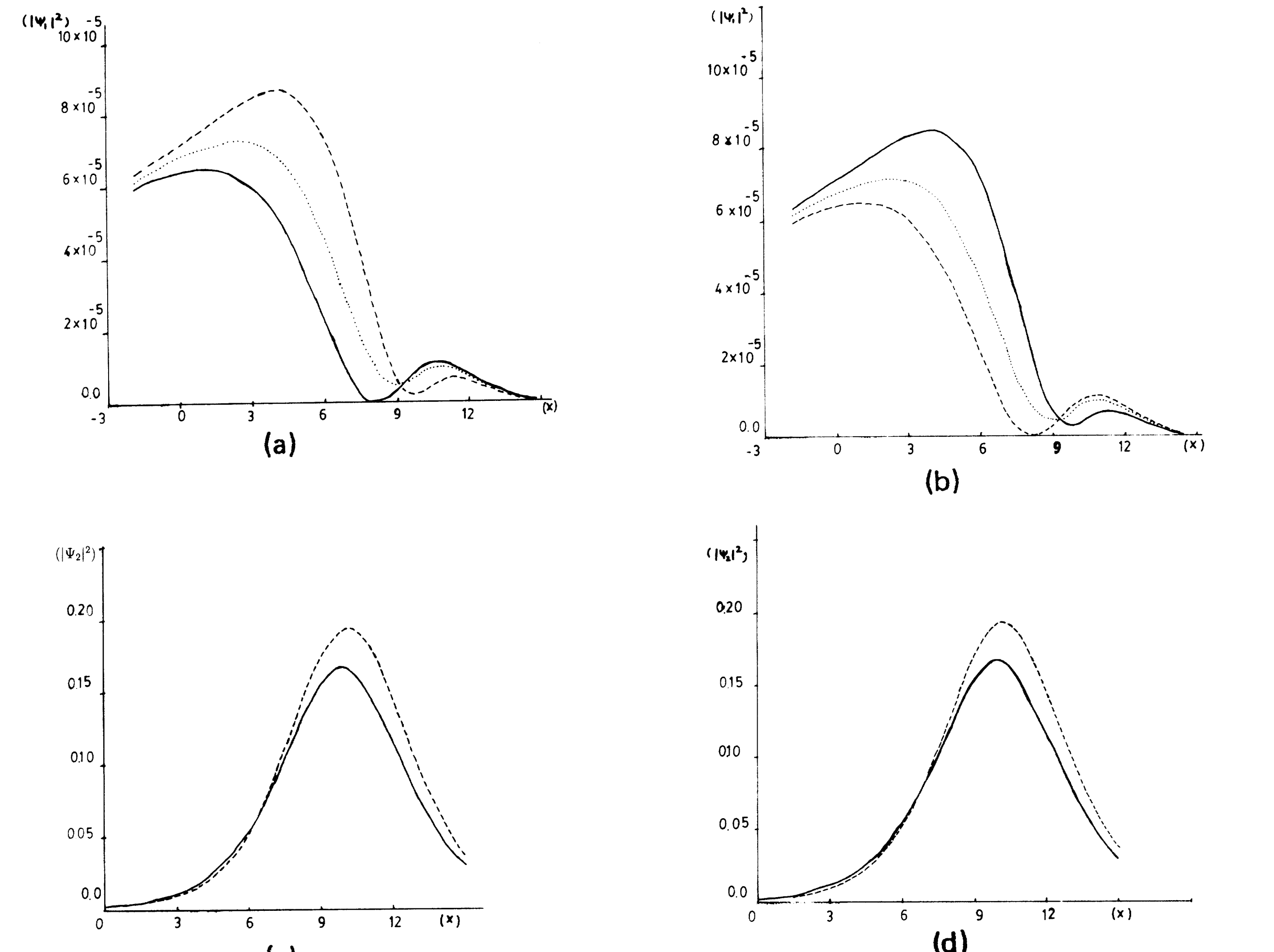

FIG. 9. For the values of $\varepsilon=0.05$, $x_{10}=0.9$, $x'_{10}=0.0$, $p_1=0.45$, $p_2=0.55$, $q_1=-0.45$, $q_2=-\frac{5}{9}$, $q=-1$, $b_1=0.03$, $b_2=0.3$, $a_1=0.03$, $a_2=0.03$, (a) $|\psi_1|^2$ is plotted against x for the solution (29): ——, $t=0$; $\cdot\cdot\cdot\cdot$, $t=30$; — — —, $t=50$, (b) as in (a): ——, $t=80$; $\cdot\cdot\cdot\cdot$, $t=100$; — — —, $t=130$, (c) $|\psi_2|^2$ is plotted against x for the solution (30): ——, $t=0$; $\cdot\cdot\cdot\cdot$, $t=30$; — — —, $t=50$, (d) as in (c): — — —, $t=80$; ——, $t=130$.

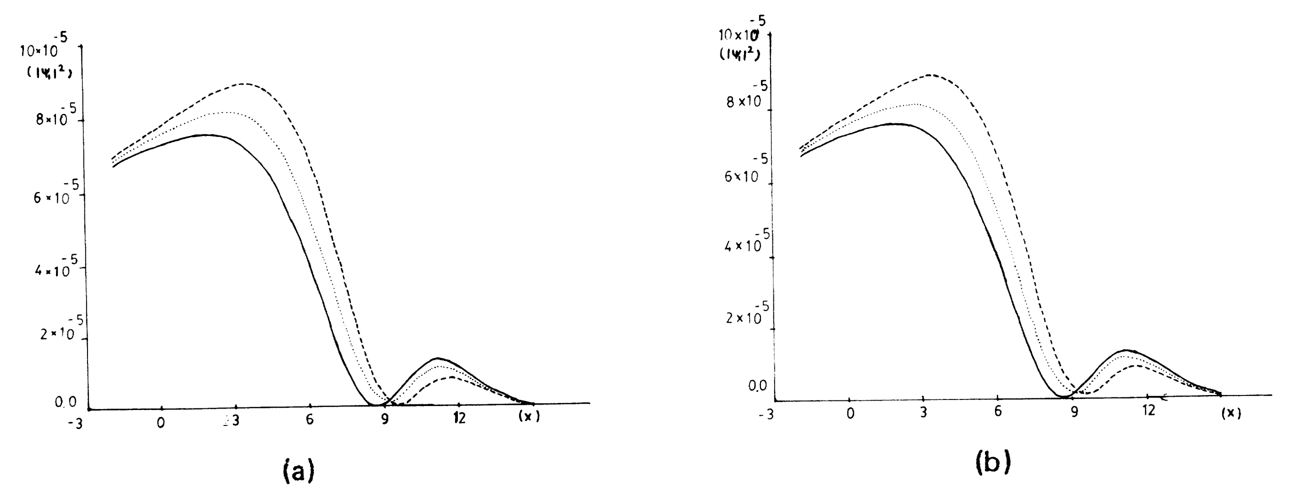

FIG. 10. With the values of $\varepsilon=0.05$, $x_{10}=0.9$, $x'_{10}=0.0$, $p_1=0.5$, $p_2=0.55$, $q_1=-0.45$, $q_2=-\frac{5}{9}$, $q=-1$, $a_1=0.03$, $a_2=0.03$, $b_1=0.03$, $b_2=0.3$, (a) $|\psi_1|^2$ is plotted against x: ——, $t=0$; $\cdot\cdot\cdot\cdot$, $t=30$; — — —, $t=50$, (b) as in (a): — — —, $t=80$; $\cdot\cdot\cdot\cdot$, $t=100$; ——, $t=130$, (c) $|\psi_2|^2$ is plotted against x: ——, $t=0$; $\cdot\cdot\cdot\cdot$, $t=50$, (d) as in (c): — — —, $t=80$; ——, $t=130$.

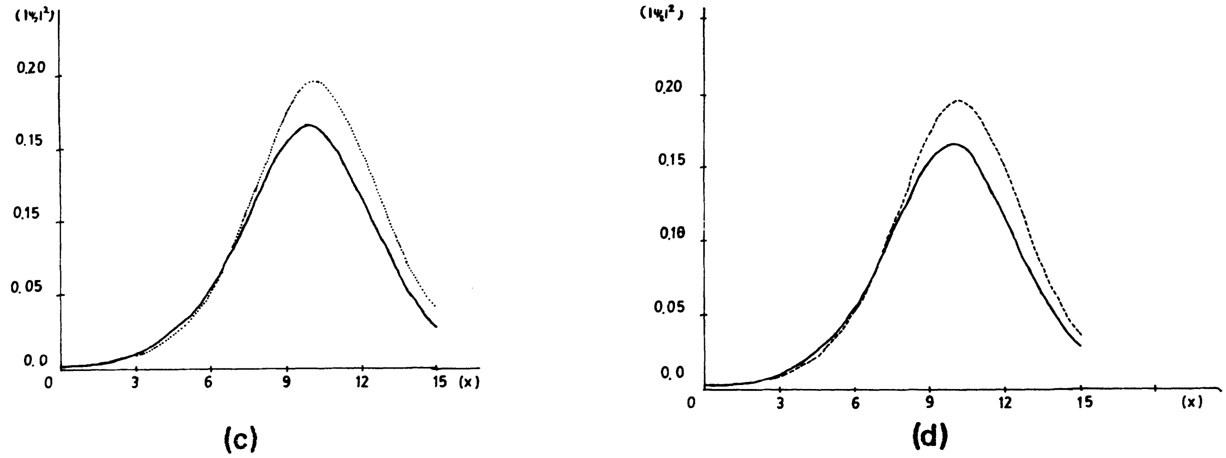

FIG. 10. (*Continued*).

change of dispersion coefficients are shown in Figs. 9 and 10, respectively.

VI. SUMMARY

The pair of coupled nonlinear Schrödinger equations (1) can be exactly solvable for the restricted cases (2) [15,16]. Interacting solitary wave solutions were known for (2a) [17]. In this paper we have extended the later case for arbitrary coefficients p_1, p_2, q_1, q_2, and q to find approximate interacting solitary wave solutions. Previous results have been regained from our solutions. By using (3) in (29) and (30) we can get other solutions of ψ_1 and ψ_2 that are traveling with velocity v. It has been shown that if we take solutions of (11) and (12) as in (47), then $|\psi_1|$ and $|\psi_2|$ can be made time independent. If solutions for $g^{(1)}$ and $h^{(1)}$ of Eqs. (11) and (12) have been taken as in (13)–(15) or (51), then we have seen that $|\psi_1|$ and $|\psi_2|$ are functions of both space and time. Approximate periodic solutions of $|\psi_1|$ and $|\psi_2|$ may be marked from Figs. 5, 6, 9, and 10. From Figs. 3 and 8 we can see that if we change the coefficient q_1 taking q_2 unchanged then $|\psi_2^{(m)}|$ is reduced for large values of q_1. If we change p_1 and p_2 then its effects may be seen from Figs. 2 and 10. We stated that if p_1 is fixed then $|\psi_2^{(m)}|$ has larger values for larger values of p_2. Again, if we take p_2 as fixed then $|\psi_2^{(m)}|$ has larger values for smaller values of p_1. If $g^{(1)}$ and $h^{(1)}$ have been taken as in (47), then for $\varepsilon=0.05$, $x_{10}=0.9=x'_{10}$, $p_1=0.45$, $p_2=0.55$, $q_1=-0.45$, $q_2=-\frac{5}{9}$, $q=-1$, $a_1=\frac{1}{3}$, $b_1=0.3$ we may have from Fig. 1(c) that $|\psi_1^{(m)}|^2\approx 3.59\times 10^{-2}$; $|\psi_2^{(m)}|^2\approx 7.56\times 10^{-2}$ and these values attained at x are nearly equal to 7.0 and 7.3, respectively. On the other hand, if we set $q=0$ we may find from (50) that $|\psi_1^{(m)}|^2\approx 0.222$, $|\psi_2^{(m)}|^2\approx 0.18$, and these values attained for x are equal to 9.9 and 10.9, respectively. From these, we may conclude that if $q<0$, then maxima of $|\psi_1|$ and $|\psi_2|$ are reduced from that in the decoupled limit.

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